THE CONDITION THAT A LINEAR TOTAL DIFFERENTIAL EQUATION BE INTEGRABLE.

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IN order that the linear partial differential equations

$$(1)\qquad \Omega_1 f \equiv \xi_1(x, y, z)\frac{\partial f}{\partial x} + \eta_1(x, y, z)\frac{\partial f}{\partial y} + \zeta_1(x, y, z)\frac{\partial f}{\partial z} = 0,$$

$$(2)\qquad \Omega_2 f \equiv \xi_2(x, y, z)\frac{\partial f}{\partial x} + \eta_2(x, y, z)\frac{\partial f}{\partial y} + \zeta_2(x, y, z)\frac{\partial f}{\partial z} = 0,$$

have a common solution it is both necessary and sufficient that a relation of the form

$$(\Omega_1, \Omega_2) \equiv \Omega_1(\Omega_2 f) - \Omega_2(\Omega_1 f) = \lambda_1(x, y, z)\Omega_1 f + \lambda_2(x, y, z)\Omega_2 f$$

exist identically, whatever the function $f(x, y, z)$ may be.*

If the above equations have the forms

$$(3)\qquad X_1 f \equiv \frac{\partial f}{\partial x} + \theta_1(x, y, z)\frac{\partial f}{\partial z} = 0,$$

$$(4)\qquad X_2 \equiv f\frac{\partial f}{\partial y} + \theta_2(x, y, z)\frac{\partial f}{\partial z} = 0,$$

the expression (X_1, X_2) will have no terms in $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, and λ_1, λ_2 accordingly become zero, that is

$$(X_1, X_2) \equiv X_1(X_2 f) - X_2(X_1 f) = 0,$$

is the necessary and sufficient condition that (3) and (4) have a common solution.

In order that the linear total differential equation

$$(5)\qquad P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz = 0$$

* See Lie-Scheffers, *Differentialgleichungen mit bekannten infinitesimalen Transformationen*, chap. 10, § 3; Leipzig, 1891.

have an integral it is both necessary and sufficient that the linear partial differential equations

$$Uf \equiv \frac{\partial f}{\partial x} - \frac{P}{R}\frac{\partial f}{\partial z} = 0, \qquad Vf \equiv \frac{\partial f}{\partial y} - \frac{Q}{R}\frac{\partial f}{\partial z} = 0$$

have a common solution.

Hence in order that the linear total differential equation (5) be integrable it is both necessary and sufficient that

$$(U,\ V) = 0;$$

or, in developed form

$$\frac{1}{R^2}\left\{ P(Q_z - R_y) + Q(R_x - P_z) + R(P_y - Q_x) \right\} \frac{\partial f}{\partial z} = 0;$$

that is

$$P(Q_z - R_y) + Q(R_x - P_z) + R(P_y - Q_x) = 0.$$

The above method admits of immediate extension to the case of more than three independent variables

$$\sum_1^n P_i(x_1, x_2, \ldots, x_n)\,dx_i = 0.$$

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